Monday is a holiday (no class, no MSC)!

## *9.5/9.6 Product, Quotient, Chain rules*

Consider the three functions:

$$y = (x^5 + 4x + 7)(x^4 + 2x) = f(x)g(x)$$ PRODUCT

$f(x) = x^5 + 4x + 7$ = First

$g(x) = x^4 + 2x$ = Second

$f'(x) = 5x^4 + 4$

$g'(x) = 4x^3 + 2$

$$y = \frac{x^4 + 5x}{x^7 - x^2} = \frac{f(x)}{g(x)}$$ QUOTIENT

$f(x) = x^4 + 5x$ = NUMERATOR

$g(x) = x^7 - x^2$ = DENOMINATOR

$f'(x) = 4x^3 + 5$

$g'(x) = 7x^6 - 2x$

$$y = (4x^2 - 3x)^{10} = f(g(x))$$ COMPOSITION

$f(u) = u^{10} \Rightarrow f'(u) = 10u^9$

$u = g(x) = 4x^2 - 3x \Rightarrow g'(x) = 8x - 3$

THE PRODUCT, QUOTIENT, AND CHAIN RULES TELL US WHAT TO DO NEXT.

**PRODUCT RULE:** $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x).$

$\quad\quad\quad\quad\quad\quad F \quad S \quad\quad F \quad S' \;+\; F' \quad S$

Ex) $y = \underbrace{(x^5+4x+7)}_{F}\underbrace{(x^4+2x)}_{S} \Rightarrow y' = (x^5+4x+7)(4x^3+2) + (5x^4+4)(x^4+2x)$

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Ex) $y = \underbrace{x^3}_{F}\underbrace{(x^2+x^{10})}_{S} \Rightarrow y' = \overset{F}{x^3}\overset{S'}{(2x+10x^9)} + \overset{F'}{3x^2}\overset{S}{(x^2+x^{10})}$

$= 2x^4 + 10x^{12} + 3x^4 + 3x^{12} = 5x^4 + 13x^{12}$ ✓

OR, EXPAND FIRST

$y = x^3x^2 + x^3x^{10} = x^5 + x^{13} \Rightarrow y' = 5x^4 + 13x^{12}$ ✓

*proof of product rule*
*(just for your own interest)*

We are trying to find a pattern for

$$\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

Adding and subtracting $f(x+h)g(x)$ in the numerator gives

$$\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

Then rearranging gives

$$=\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

$$=f(x+h)\frac{g(x+h)-g(x)}{h}+\frac{f(x+h)-f(x)}{h}\,g(x)$$

As $h \to 0$, we see the expression above is approaching
$f(x)\,g'(x)+f'(x)\,g(x)$

**QUOTIENT RULE:**

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$\left(\frac{N}{D}\right)' = \frac{DN' - ND'}{D^2}$$

Ex) $y = \dfrac{x^4+5x}{x^7-x^2} \Rightarrow y' = \dfrac{(x^7-x^2)(4x^3+5) - (x^4+5x)(7x^6-2x)}{(x^7-x^2)^2}$

Ex) $y = \dfrac{4x+2}{x^3} \Rightarrow y' = \dfrac{x^3(4) - (4x+2)3x^2}{x^6} = \dfrac{4x^3-12x^3-6x^2}{x^6}$

$= \dfrac{-8x^3-6x^2}{x^6} = -8x^{-3}-6x^{-4}$

OR, EXPAND FIRST

$y = \dfrac{1}{x^3}(4x+2) = \dfrac{4x}{x^3} + \dfrac{2}{x^3} = 4x^{-2}+2x^{-3} \Rightarrow y' = -8x^{-3}-6x^{-4}$

$= -\dfrac{8}{x^3} - \dfrac{6}{x^4}$ ← ONLY POSITIVE EXPONENTS

You try: Differentiate

1. $y = x^2(x^3 + 1)$

EXPANDING FIRST $\Rightarrow y = x^5 + x^2$

$y' = 5x^4 + 2x$

PRODUCT RULE $\Rightarrow y' = x^2 \cdot 3x^2 + 2x(x^3+1)$

$= 3x^4 + 2x^4 + 2x$

$= 5x^4 + 2x$

2. $y = \dfrac{5}{x^3}$

REWRITE FIRST, $y = 5x^{-3}$

$y' = -15x^{-4} = -\dfrac{15}{x^4}$

$\boxed{\text{HW NOTE}}$

ANSWER WRITTEN USING ONLY POSITIVE EXPONENTS

QUOTIENT RULE

$y' = \dfrac{x^3(0) - 5 \cdot 3x^2}{x^6}$

$= \dfrac{-15x^2}{x^6} = -15x^{-4}$

3. $y = (x^2 + 3x)(\sqrt{x} - 5x^3)$

PRODUCT RULE

$$\boxed{y' = (x^2+3x)\left(\tfrac{1}{2}x^{-1/2} - 15x^2\right) + (2x+3)(x^{1/2} - 5x^3)}$$

OR $y = x^{2.5} - 5x^5 + 3x^{1.5} - 15x^4$

$y' = 2.5x^{1.5} - 25x^4 + 4.5x^{0.5} - 60x^3$

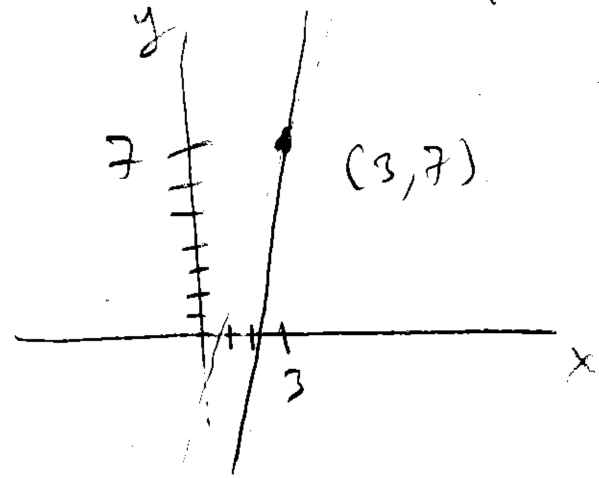
4. $y = \dfrac{x^5}{3x^3 - x^5}$

QUOTIENT RULE (ONLY OPTION)

$$\boxed{y' = \frac{(3x^3 - x^5)5x^4 - x^5(9x^2 - 5x^4)}{(3x^3 - x^5)^2}}$$

**Equations for Tangent lines**
*(HW 9.5: Problems 8 and 9)*

slope = 8




Recall: All the points $(x, y)$ on a given line can be described by an equation of the form

$$y = m(x - x_0) + y_0$$

where

$$m = \text{slope of the line}$$
$$(x_0, y_0) = \text{any point on the line}$$

means

$$\frac{y-7}{x-3} = 8$$

FOR ALL $(x,y)$ ON THE LINE.

Review Question:
Find the equation of the line that has slope 8 and goes through (3, 7).

$$\boxed{y = 8(x-3)+7}$$

$$y = m(x - x_0) + y_0$$

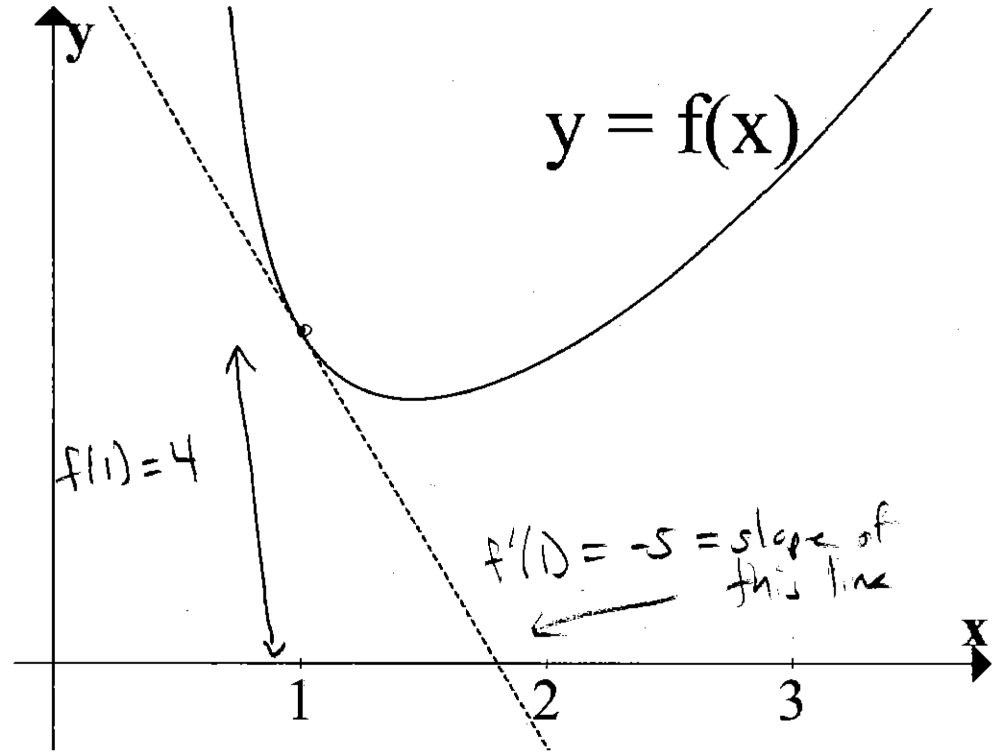
CAN ALSO BE EXPANDED TO GET

$$y = 8x - 24 + 7$$

$$\boxed{y = 8x - 17}$$

$$y = mx + b$$

Since $f'(x)$ is the slope of the tangent line, we can use it to get the equation for the tangent line.




*Example*: Let

$$f(x) = \frac{x^3+3}{2x-1}$$

Find the equation for the tangent line at $x = 1$.

$$y = f(x) = \frac{x^3+3}{2x-1} = \text{HEIGHT ON GRAPH}$$

$$y' = f'(x) = \frac{(2x-1)(3x^2) - (x^3+3)2}{(2x-1)^2} = \text{TANGENT SLOPE ON GRAPH}$$

$$f(1) = \frac{(1)^3+3}{2(1)-1} = 4$$

$$f'(1) = \frac{(2(1)-1)(3(1)^2) - ((1)^3+3)\cdot 2}{(2(1)-1)^2} = \frac{1\cdot 3 - 4\cdot 2}{1^2} = 3-8 = -5$$

$$\boxed{y = -5(x-1)+4}$$

$$y = -5x+5+4$$

$$y = -5x+9$$

# *Section 9.6: The chain rule*

**CHAIN RULE:**

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

Ex) $y = (4x^2 - 3x)^{10} = f(g(x))$

OUTSIDE $= f(u) = u^{10} \Rightarrow 10u^9$

INSIDE $= g(x) = 4x^2 - 3x \Rightarrow 8x - 3$

$y' = 10(4x^2 - 3x)^9 \cdot (8x - 3)$

Ex) $y = (6x + x^{14})^{100}$

$y' = 100(6x + x^{14})^{99} \cdot (6 + 14x^{13})$

Ex) $y = \sqrt{x^2 + 3x} = (x^2 + 3x)^{1/2}$

$y' = \frac{1}{2}(x^2 + 3x)^{-1/2}(2x + 3) = \frac{1}{2} \frac{1}{(x^2+3x)^{1/2}} 2x + 3 = \frac{2x + 3}{2(x^2 + 3x)^{1/2}}$

*proof of chain rule*
*(just for your own interest)*

We are trying to find a pattern for

$$\frac{f(g(x+h)) - f(g(x))}{h}$$

Multiplying top and bottom by

$g(x+h) - g(x)$ gives

$$\left(\frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{h}\right)\left(\frac{g(x+h) - g(x)}{g(x+h) - g(x)}\right)$$

Rearranging gives

$$\left(\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\right)\left(\frac{g(x+h) - g(x)}{h}\right)$$

As $h \to 0$, we see the expression above is approaching

$$f'\big(g(x)\big)g'(x)$$

***All Rules:***

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x).$$

$$\frac{d}{dx}\left(cf(x)\right) = cf'(x).$$

$$\frac{d}{dx}(x^n) = n\, x^{n-1}.$$

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x).$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$\frac{d}{dx}(f(g(x))) = f'\left(g(x)\right)g'(x)$$